y, z) = (0, 0, 0.3λ). The saving in computation time in using the θ-algorithm over a direct summation of the series is of the order of 100.

V. Conclusion

The use of θ-algorithm is shown to have a dramatic impact in accelerating the convergence of slowly converging series. The algorithm has been applied with success to the series representing the free-space periodic Green's functions. Numerical results indicate that the algorithm is superior to Shanks' transform both in convergence and speed. In most cases the algorithm converges to a high degree of precision in about 20 terms. This is indeed remarkable as a direct sum of the series converges extremely slowly. The use of θ-algorithm results in a considerable amount of saving in computation time thereby increasing the computational efficiency in problem involving one-dimensional periodicity.

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On the Use of Chebyschev–Toeplitz Algorithm in Accelerating the Numerical Convergence of Infinite Series

Surendra Singh and Ritu Singh

Abstract—**It is shown here that a simple application of the Chebyschev–Toeplitz algorithm enhances the rate of convergence of slowly converging series. The algorithm is applied to series representing the periodic Green's functions involving a single infinite summation. The algorithm yields highly accurate results within relatively fewer terms. A quantitative comparison is shown with methods previously reported in the literature.**

I. Introduction

The computation of electromagnetic radiation or scattering from a periodic geometry involves the summation of a Green's function series which converges very slowly. The summation of the series may be accelerated by transforming the series such that the new series converges rapidly [1]–[5]. The transformation, however, requires analytical work which is characteristic for each series. This in some sense limits the applicability of the method. It is our intent to demonstrate that algorithms [6]–[11] which can be readily applied to any slowly converging series, irrespective of its functional form, are highly accurate and efficient. In particular, we report the use of the Chebyschev–Toeplitz (CT) algorithm [11] in accelerating the convergence of periodic Green's function series.

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II. Chebyschev–Toeplitz (CT) Algorithm

Let S_n be the partial sum of n terms of a series such that $S_n \to S$ as $n \to \infty$, where S is the sum of the series. The CT algorithm is defined by the following equations [11]:

$$t_{-1}^{(n)} = 0, \quad t_0^{(n)} = S_n, \quad \sigma_0 = 1, \tag{1}$$

$$t_1^{(n)} = t_0^{(n)} + 2t_0^{(n+1)}, \quad \sigma_1 = 3, \tag{2}$$

$$t_{k+1}^{(n)} = 2t_k^{(n)} + 4t_k^{(n+1)} - t_{k-1}^{(n)}, \qquad k = 1, 2, \cdots \tag{3}$$

$$\sigma_{k+1} = 6\sigma_k - \sigma_{k-1}, \qquad k = 1, 2, \cdots \tag{4}$$

$$T_k^{(n)} = t_k^{(n)}/\sigma_k, \qquad k = 0, 1, 2, \cdots . \tag{5}$$

The nth iterate of the CT algorithm is given by $T_k^{(n)}$, which gives an estimate of the sum of the series. The algorithm can be illustrated by applying it to the slowly converging Liebniz series for π:

$$\pi = \sum_{m=0}^{\infty} \frac{4(-1)^m}{2m+1}. \tag{6}$$

The result of applying the CT algorithm to the sequence of partial sums $S_0, S_1, \cdots, S_8$ is given in Table I. The algorithm converges to six significant digits. Although the even and odd order iterates of the CT algorithm provide an estimate of S, only the even orders are shown in the table.

III. Free-Space Periodic Green's Functions

The spectral domain Green's function for a one-dimensional array of line source spaced d units apart in the x direction is given by

$$G = \sum_{m=-\infty}^{\infty} \frac{1}{j2dk_{y_m}} \cdot \exp(-jk_{y_m}|y-y'|\exp[-j(2m\pi/d)(x-x')] \tag{7}$$

where

$$k_{y_m} = \begin{cases} \sqrt{k^2 - (2m\pi/d)^2}, & k^2 > (2m\pi/d)^2 \\ -j\sqrt{(2m\pi/d)^2 - k^2}, & k^2 < (2m\pi/d)^2 \end{cases}$$

k is the wave number of the medium, (x', y') locates the reference source and (x, y) locates the observation point. The series in (7) converges very slowly whenever $y = y'$. This is referred to as the "on plane" case. The spatial domain counterpart of the periodic Green's function in (7) is given by

$$G = \sum_{m=-\infty}^{\infty} H_0^{(2)}(k[(y-y')^2 + (x-x'-md)^2]^{1/2}) \tag{8}$$

where $H_0^{(2)}$ is the zeroth-order Hankel function of the second kind.

The Green's function for a one-dimensional array of point sources

TABLE I
APPLICATION OF CT ALGORITHM TO THE LEIBNITZ SERIES FOR π

n	$T_0^{(n)} = S_n$	$T_2^{(n)}$	$T_4^{(n)}$	$T_6^{(n)}$	$T_8^{(n)}$
0	4.0000000	3.1215687	3.1415255	3.1415923	3.1415930
1	2.6666667	3.1507003	3.1416481	3.1415937	
2	3.4666667	3.1380019	3.1415079	3.1415923	
3	2.8952381	3.1424160	3.1416895		
4	3.3396825	3.1422331	3.1414986		
5	2.9760462	3.1401489			
6	3.2837385	3.1434848			
7	3.0170718				
8	3.2523659				

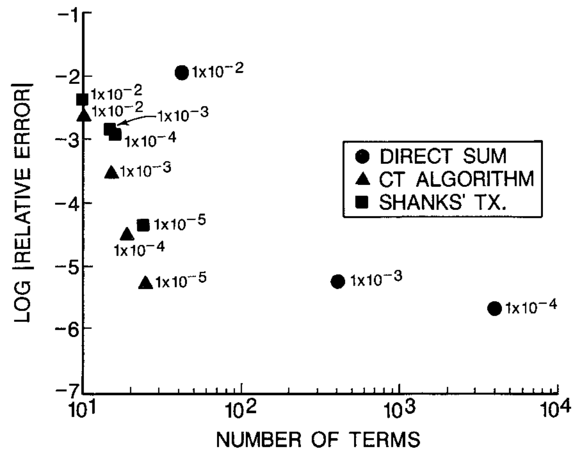

Fig. 1. Log of relative error magnitude versus number of terms for the periodic Green's series in (7) for $d = 0.5\lambda$ and $(x, y) = (0.3\lambda, 0.0\lambda)$.

located d units apart in the z direction is given by

$$G = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \cdot \frac{\exp\{-jk[(x-x')^2 + (y-y')^2 + (z-md)^2]^{1/2}\}}{[(x-x')^2 + (y-y')^2 + (z-md)^2]^{1/2}}. \quad (9)$$

The above series converges extremely slowly for all combinations of source and observation points. In the following section we present the results of applying the CT algorithm to the series in (7), (8), and (9).

IV. NUMERICAL RESULTS

For the numerical computation of the periodic Green's functions in (7), (8), and (9), we take $(x', y') = (0, 0)$ and $\lambda = 1.0$ m. Results obtained from a direct summation of the series and the application of Shanks' transform are shown for comparison. Figs. 1-3 show the logarithm of the magnitude of a relative error measure as a function of the number of terms (or partial sums for the CT algorithm and Shanks' transform) for different values of a convergence factor [1], ϵ_c, for the series in (7). The CT algorithm converges faster with a much lower error than both the direct sum and the Shanks' transform. Since $y = y'$ in all cases, representing the "on plane" case, the series has the slowest convergence. Even then, for $(x, y) = (0.6\lambda, 0)$, the CT algorithm converges to machine precision (zero relative error) in merely 17 terms for $\epsilon_c = 1 \times 10^{-5}$, as shown in Fig. 2. In Figs. 4-6 is shown the relative error measure for the spatial domain periodic Green's function given in (8). The CT algorithm converges within 30 terms to five or more significant digit accuracy. It is worth pointing out that for the cases shown in Figs. 4 and 6, the relative error for the CT algorithm goes to zero for $\epsilon_c = 1 \times 10^{-6}$. This signifies that the algorithm has converged to machine precision.

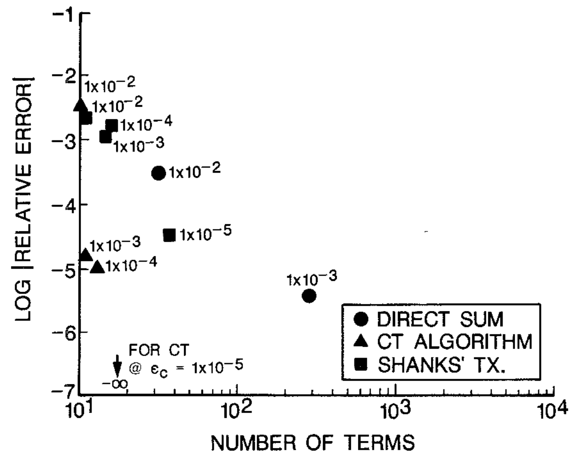

Fig. 2. Log of relative error magnitude versus number of terms for the periodic Green's function series in (7) for $d = 1.2\lambda$ and $(x, y) = (0.6\lambda, 0.0\lambda)$.

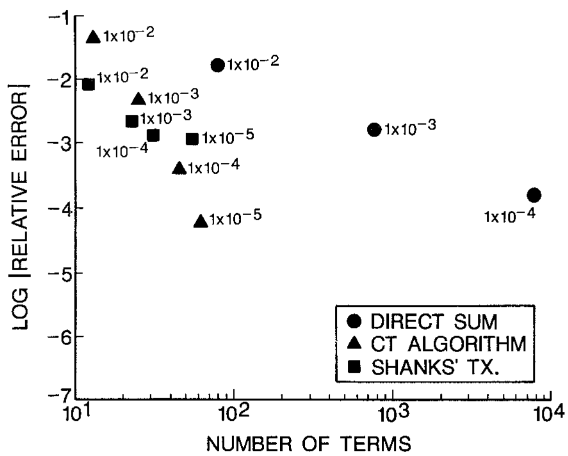

Fig. 3. Log of relative error magnitude versus number of terms for the periodic Green's function series in (7) for $d = 1.2\lambda$ and $(x, y) = (0.3\lambda, 0.0\lambda)$.

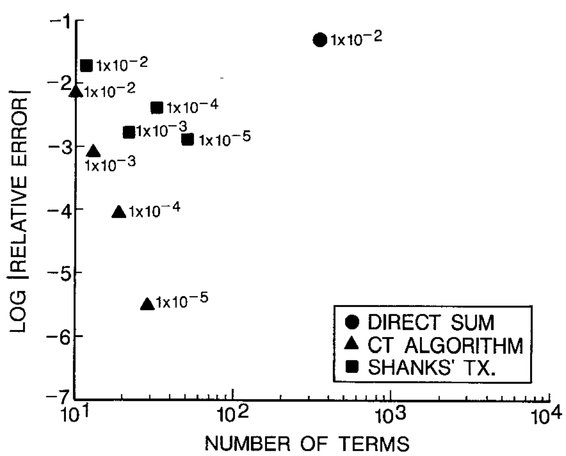

Fig. 4. Log of relative error magnitude versus number of terms for the periodic Green's function series in (8) for $d = 0.6\lambda$ and $(x, y) = (0.3\lambda, 0.1\lambda)$.

Figs. 7 and 8 show the relative error magnitude as a function of the number of terms for the periodic Green's function series in (9) for various locations of the observation point. In each case, the CT algorithm converges in less than 25 terms with the lowest error. A direct summation of the series takes several thousand terms to converge to four significant digits. The Shanks' transform converges in less than 55 terms but the error does not improve as the conver-

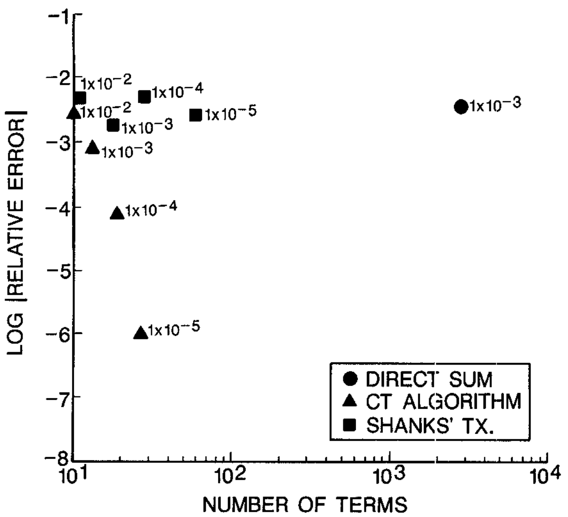

Fig. 5. Log of relative error magnitude versus number of terms for the periodic Green's function series in (8) for $d = 0.6\lambda$ and $(x, y) = (0.3\lambda, 0.01\lambda)$.

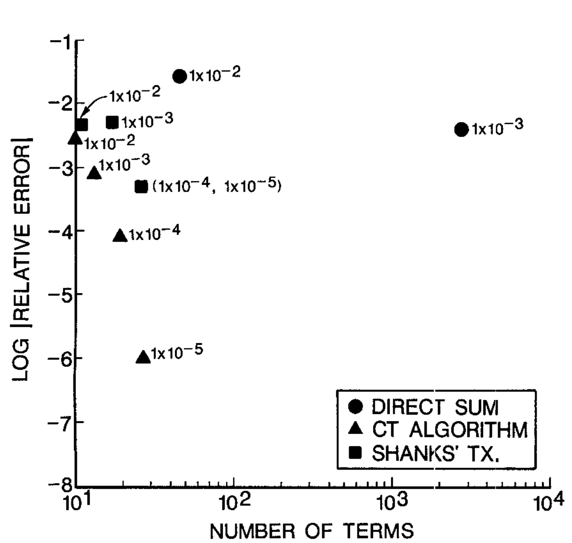

Fig. 6. Log of relative error magnitude versus number of terms for the periodic Green's function series in (8) for $d = 0.6\lambda$ and $(x, y) = (0.3\lambda, 0.0\lambda)$.

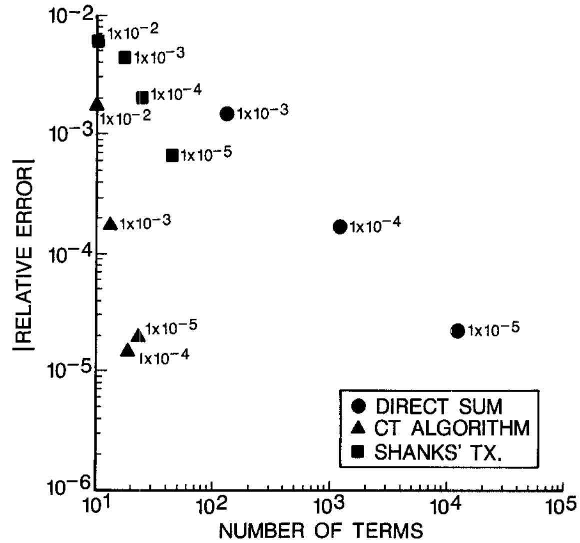

Fig. 7. Relative error magnitude versus number of terms for the periodic Green's function series in (9) for $d = 0.6\lambda$ and $(x, y, z) = (0.2\lambda, 0.1\lambda, 0.3\lambda)$.

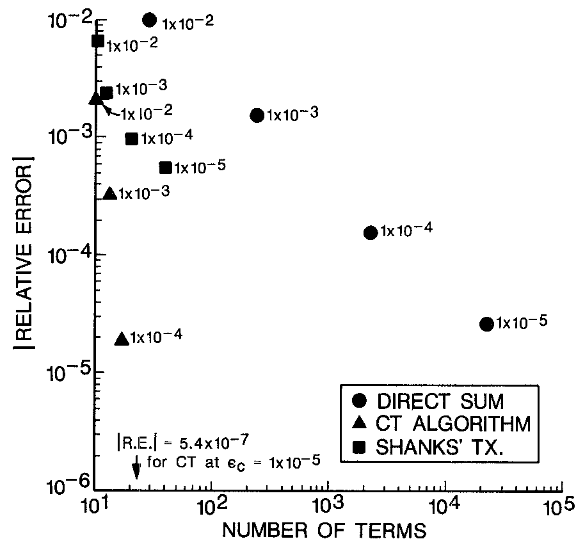

Fig. 8. Relative error magnitude versus number of terms for the periodic Green's function series in (9) for $d = 0.6\lambda$ and $(x, y, z) = (0.1\lambda, 0.1\lambda, 0.3\lambda)$.

gence factor is lowered. It is shown in Fig. 8, for $(x, y) = (0.1\lambda, 0.1\lambda, 0.3\lambda)$, that the CT algorithm converges close to seven significant digits for $\epsilon_c = 1 \times 10^{-5}$.

V. Conclusion

We have demonstrated that the use of CT algorithm enhances the convergence of infinite series. This is shown by the application of CT algorithm to three slowly converging periodic Green's function series involving a single infinite summation. The CT algorithm performed superior in speed and convergence in comparison to the Shanks' transform and the direct sum.

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